THE PARAXIAL APPROXIMATION FOR A DENSE ELECTRON BEAM

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Approximate equations are derived for a narrow (paraxial) electron beam with a three-dimensional axis; these equations generalize the familiar equations [1] to the case in which the field of the charge in the beam is important. A class of beams is distinguished for which the problem reduces to ordinary differential equations.

The paraxial approximation for a narrow beam of electron trajectories in a given field is well known for the case in which the characteristic dimension L_* of the irregularities is much greater than a_*, the characteristic width of the beam. The field of a three-dimensional beam can be included in the equations [1] if the beam density ρ is nonuniform over lengths L_*. The self-field of the beam was not actually taken into account in [1]. In this paper we attempt to remedy this deficiency, with partial success for an axisymmetric beam with a rectilinear axis [2].

1. The law of plane cross sections. It is convenient to use a form for the equations of a monoenergetic beam in which the Clebsch variables ξ, ζ, χ are used to represent the velocity field of the electrons v_α:

$$g^{\alpha\beta}v_\beta\xi_{,\alpha} = g^{\alpha\beta}v_\beta\zeta_{,\alpha} = (\rho g g^{\alpha\beta}v_\beta)_{,\alpha} = 0,$$
$$v_\alpha \equiv A_\alpha + \xi\zeta_{,\alpha} + \chi_{,\alpha}. \tag{1.1}$$

Here q^α are curvilinear coordinates with a metric tensor $g^{\alpha\beta}$; the subscript following the comma denotes a derivative with respect to the appropriate coordinate; subscripts which appear twice are summed from 1 to 3; A_α are the covariant components of the potential of the magnetic field H_α. The remaining equations for a steady nonrelativistic beam have the form

$$g^{\alpha\beta}v_\alpha v_\beta = 2\varphi, \quad (g g^{\alpha\beta}\varphi_{,\alpha})_{,\beta} = \rho, \quad g^{-2} \equiv \det|g^{\alpha\beta}|. \tag{1.2}$$

Here φ is the potential of the electric field. The physical constants ($e > 0$ is the charge and m the mass of an electron; c is the velocity of light) are omitted, which corresponds to the following change of symbols

$$e\,(mc)^{-1}A_\alpha \to A_\alpha, \quad e\,(mc)^{-1}H_\alpha \to H_\alpha,$$
$$(e\,/\,m)\varphi \to \varphi, \quad 4\pi\,(e\,/\,m)\rho \to \rho.$$

Here $l \equiv q^1$ is the longitudinal coordinate and $s \equiv q^2$ and $q \equiv q^3$ are transverse coordinates relative to the axial line $R(l)$ of the narrow beam associated with the Cartesian r:

$$\mathbf{r} = \mathbf{R} + \mathbf{s}s + \mathbf{q}q, \quad \mathbf{s} \equiv \mathbf{l}'\,/\,k,$$
$$\mathbf{l} \equiv \mathbf{R}', \quad \mathbf{q} \equiv \mathbf{l} \times \mathbf{s}, \quad \mathbf{R}' \equiv d\mathbf{R}\,/\,dl. \tag{1.3}$$

Here l is the arc length of the axis, and the unit vectors l, s, and q form the accompanying right-handed set (Fig. 1). Using the Frenet [3] formulas we have

$$\mathbf{s}' = -k\mathbf{l} + \varkappa\mathbf{q}, \quad \mathbf{q}' = -\varkappa\mathbf{s},$$
$$k \equiv |\mathbf{l}'|, \quad \varkappa k^2 \equiv (\mathbf{l} \times \mathbf{l}')\,\mathbf{l}''. \tag{1.4}$$

Here k is the curvature, $\varkappa$ is the torsion of the axis, and it can easily be shown that

$$dr^2 = \sigma^2 dl^2 + ds^2 + dq^2 - 2\varkappa q\,dl\,ds + 2\varkappa s\,dl\,dq, \tag{1.5}$$

$$g_{\alpha\beta} = \begin{vmatrix} \sigma^2 & -\varkappa q & \varkappa s \\ -\varkappa q & 1 & 0 \\ \varkappa s & 0 & 1 \end{vmatrix},$$
$$g^{\alpha\beta} = \frac{1}{g^2}\begin{vmatrix} 1 & \varkappa q & -\varkappa s \\ \varkappa q & \sigma^2 - \varkappa^2 s^2 & -\varkappa^2 sq \\ -\varkappa s & -\varkappa^2 sq & \sigma^2 - \varkappa^2 q^2 \end{vmatrix}. \tag{1.6}$$

In relationships (1.6)

$$g^2 \equiv \det|g_{\alpha\beta}| = (1 - ks)^2,$$
$$\sigma^2 \equiv (1 - ks)^2 + \varkappa^2(s^2 + q^2).$$

In (1.1), (1.2), and (1.6) an index of smallness ε should be set in those places where the small parameter ε_* appears as the result of transition to the dimensionless quantities

$$s\,/\,a_*, \; q\,/\,a_*, \; l\,/\,L_*, \; \varkappa L_*, \; kL_*, \; \varepsilon_* \equiv a_*\,/\,L_*,$$

i.e., in front of k, $\varkappa$, and derivatives with respect to l. In doing so we single out the case of a narrow beam close to a relatively smooth axis. As an example, the equations for A_α in a nonrelativistic beam can be written in the form

$$A_{q,s} - A_{s,q} = gH^l, \quad A_{l,q} - \varepsilon A_{q,l} = gH^s,$$
$$\varepsilon A_{s,l} - A_{l,s} = gH^q, \quad H_{q,s} = H_{s,q},$$
$$H_{l,q} = \varepsilon H_{q,l}, \quad H_{l,s} = \varepsilon H_{s,l}, \quad H_\alpha \equiv g_{\alpha\beta}H^\beta. \tag{1.7}$$

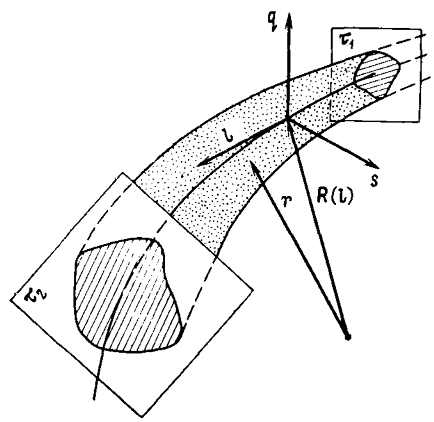

Fig. 1

This leads to the paraxial approximation for A_α:

$$A_l = \Omega_s q - \Omega_q s + \varepsilon^{1/2}(k\Omega_q + 2\varkappa\Omega_l)s^2,$$
$$A_s = -{}^1/_2\Omega_l q, \quad A_q = {}^1/_2\Omega_l s. \tag{1.8}$$

Here Ω_l, Ω_s, and Ω_q are the values of H_l, H_s, and H_q on the axis. In a paraxial beam the longitudinal velocity v_l compared with the transverse velocity v_s, v_q and the potential φ compared with v_s^2 and v_q^2 must be represented in the form

$$v_l = \varepsilon^{-1}v(l) + A_l + \varepsilon P_l,$$
$$\varphi = \varepsilon^{-2}U + \varepsilon^{-1}(E_s s + E_q q) + \Phi. \tag{1.9}$$

Keeping the above in mind we easily obtain from (1.1), (1.2), (1.6), (1.8), and (1.9) that

$$2U = v^2, \quad E_s = kv^2 - v\Omega_q, \quad E_q = v\Omega_s,$$
$$\xi_{,\tau} + u_s\xi_{,s} + u_q\xi_{,q} = \zeta_{,\tau} + u_s\zeta_{,s} + u_q\zeta_{,q} = 0,$$
$$\xi_{,\tau} \equiv (\partial\xi/\partial l)v, \tag{1.10}$$

$$(\rho v)_{,\tau} + (u_s\rho v)_{,s} + (u_q\rho v)_{,q} = 0,$$
$$u_{s(q)} \equiv B_{s(q)} + \xi\zeta_{,s(q)} + \chi_{,s(q)}, \tag{1.11}$$

$$u_s^2 + u_q^2 + 2(\xi\zeta_{,\tau} + \chi_{,\tau}) = 2\psi,$$
$$\psi_{,ss} + \psi_{,qq} = \rho - n, \quad dl \equiv v d\tau,$$
$$B_s \equiv -{}^1/_2\Omega q, \quad B_q \equiv {}^1/_2\Omega s, \quad \Omega \equiv \Omega_l - 2\varkappa v,$$
$$2\Phi \equiv 2\psi + v^2(3k^2s^2 - \varkappa^2 s^2 - \varkappa^2 q^2) +$$
$$+ (\Omega_s q - \Omega_q s)^2 + vs(4k\Omega_s q - 3\Omega_q ks + 2\Omega_l\varkappa s),$$
$$n \equiv 2(k^2 - \varkappa^2)v^2 + \Omega_s^2 + \Omega_q^2 +$$
$$+ 2v(\varkappa\Omega_l - k\Omega_q) + U'', \tag{1.12}$$

Here Eqs. (1.10) correspond to the zeroth and first approximations in ε, and (1.11) to the second approximation. If we assume that U is the potential and E_s and E_q are the field strengths on the axis, then Eqs. (1.10) are the exact equations for the electron trajectories in the field of the beam. The sense of Eqs. (1.11) appears more clearly when they are written in the Lagrange variables τ, ξ, and η, where ξ and η are the coordinates of the electron in the initial cross section $\tau = 0$:

$$s_{,\tau\tau} = e_s - \Omega q_{,\tau},$$
$$q_{,\tau\tau} = e_q + \Omega s_{,\tau}, \quad \Omega \equiv \Omega_l - 2\varkappa v, \tag{1.13}$$

$$v\rho = j(\xi, \eta)\, |s_{,\xi}q_{,\eta} - s_{,\eta}q_{,\xi}|^{-1},$$
$$s = s(\tau, \xi, \eta), \quad q = q(\tau, \xi, \eta), \tag{1.14}$$

$$e_{s,s} + e_{q,q} = \rho - n,$$
$$e_{q,s} - e_{s,q} = \dot\Omega, \quad \dot\Omega \equiv d\Omega / d\tau. \tag{1.15}$$

Here j is the current density in the initial cross section, the dot indicates a derivative with respect to τ, and a dash indicates a derivative with respect to l. These equations are equivalent to the equations of a nonstationary plane (s, q) electron cloud on a "space charge" background n (τ) which is alternating in "time" τ in a uniform "magnetic field" Ω but with one difference: the density, multiplied by a function of time, satisfies the equation of continuity (ρv is a longitudinal component of current density of the beam). We thus obtain a generalization of the familiar law of plane cross sections [4] for the case of curvilinear motion with a varying axial velocity. As one would expect, nonstationarity of the magnetic field leads to the appearance of the curl $\dot\Omega$ in the electric field strength e_s, e_q.

It should be noted that the following can be reduced to Eqs. (1.11): (1) the equations of a nonenergetic beam with an electron energy $\mathscr{E}(\xi)$, (2) the equations of a nonsteady (t) beam, if we assume that the beam parameters in (1.11) may be arbitrary functions of $t - \tau$; i.e., nonsteady perturbations in the narrow beam propagate in the form of waves moving along the beam with a velocity v.

The initial equations for a relativistic beam differ only in the form of (1.2),

$$g^{\alpha\beta}\, v_\alpha v_\beta = 2\varphi + (\varphi / c)^2, \quad (gg^{\alpha\beta}\,\varphi_{,\alpha})_{,\beta} = \rho(1 + \varphi c^{-2}), \tag{1.16}$$

if we assume that v_α/c is the 4-velocity and ρ is the scalar charge density. Let v_l be on the order of c. Then representing v_l and φ in the form (1.9) with an accuracy to ε^2, we easily obtain the same equations as in (1.11) and also

$$2U + (U / c)^2 = v^2, \quad \gamma E_s = kv^2 - v\Omega_q,$$
$$\gamma E_q = v\Omega_s, \quad \gamma \equiv 1 + Uc^{-2}. \tag{1.17}$$

The difference is in the expressions for A_l, Φ, and n:

$$2\gamma\Phi \equiv 2\psi + 2vA - (E_s s + E_q q)^2 c^{-2} +$$
$$+ v^2(3k^2s^2 - \varkappa^2 s^2 - \varkappa^2 q^2) + (\Omega_s q - \Omega_q s)^2 +$$
$$+ vs(4k\Omega_s q - 3ks\Omega_q + 2\varkappa\Omega_l s),$$
$$A_{,ss} + A_{,qq} = \rho v c^{-2},$$
$$n \equiv 2(k^2 - \varkappa^2)v^2 + 2(\varkappa\Omega_l - k\Omega_q)v +$$
$$+ \gamma U'' + \Omega_s^2 + \Omega_q^2 - (E_s^2 + E_q^2)c^{-2}. \tag{1.18}$$

Here εA is an increment to A_l in (1.8) which takes into account the self (pinching) magnetic field of the beam. Thus the law of plane cross sections in the form (1.13)–(1.15) is also valid for the case of a weakly relativistic beam.

2. Degenerate Solutions. We include uniform beam deformation in a plane cross section within the framework of the inverse problem, as was done in [5] for the case v = const and a rectilinear axis:

$$s = \alpha\xi + \beta\eta, \quad q = \mu\xi + \nu\eta, \quad D(\tau) \equiv \alpha\nu - \beta\mu. \tag{2.1}$$

In this case the axis coincides exactly with an electron trajectory. Inserting (2.1) into (1.13)–(1.15) we have

$$De_s = [\ddot\alpha\nu - \ddot\beta\mu + \Omega(\dot\mu\nu - \dot\nu\mu)]s +$$
$$[\ddot\beta\alpha - \ddot\alpha\beta + \Omega(\dot\nu\alpha - \dot\mu\beta)]q,$$
$$De_q = [\ddot\mu\nu - \ddot\nu\mu + \Omega(\dot\beta\mu - \dot\alpha\nu)]s +$$
$$[\ddot\nu\alpha - \ddot\mu\beta + \Omega(\dot\alpha\beta - \dot\beta\alpha)]q, \tag{2.2}$$

$$\dot\mu\nu - \dot\nu\mu + \dot\alpha\beta - \dot\beta\alpha = \Omega D - \omega_*, \quad \omega_* = \text{const}, \tag{2.3}$$

$$\ddot D - 2(\dot\alpha\dot\nu - \dot\beta\dot\mu) + (n + \Omega^2)D - \Omega\omega_* = j_* / v,$$
$$\rho = j_*(vD)^{-1}. \tag{2.4}$$

The same result is also obtained if we look for a solution of the initial equations in the form of a Taylor series in powers of s and q. Thus (2.1)–(2.4) describe a narrow tube of trajectories, cut out of a wide beam where the irregularity dimension is L_*. Equations

(2.3) and (2.4) leave five of the seven functions of $\tau(\alpha, \beta, \mu, \nu, v, k, \varkappa)$ arbitrary, which means that we have a fairly flexible basis for forming narrow beams with the necessary parameters. For the five simplest types of two-parameter deformations with the matrices

$$M_1 = \begin{vmatrix} D & 0 \\ \mu & 1 \end{vmatrix}, \quad M_2 = \begin{vmatrix} D & \beta \\ 0 & 1 \end{vmatrix},$$

$$M_3 = \begin{vmatrix} \delta\cos\theta & -\delta\sin\theta \\ \delta\sin\theta & \delta\cos\theta \end{vmatrix},$$

$$M_4 = \begin{vmatrix} D\cos\theta & -\sin\theta \\ D\sin\theta & \cos\theta \end{vmatrix},$$

$$M_5 = \begin{vmatrix} D\cos\theta & -D\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}, \quad M \equiv \begin{vmatrix} \alpha & \beta \\ \mu & \nu \end{vmatrix}, \tag{2.5}$$

Eqs. (2.3) and (2.4) assume the following forms, respectively:

$$\mu = \Omega D - \omega_*, (M_1), \; \dot{D}\beta - \dot{\beta} D = \Omega D - \omega_*, \; (M_2), \tag{2.6}$$

$$\ddot{D} + (n + \Omega^2) D = \Omega\omega_* + j_* / v, \; (M_1, \; M_2), \tag{2.7}$$

$$2\dot{\theta} = \Omega - \omega_* / D, (M_3, \; M_4),$$

$$\dot{\theta}(1 + D^2) = D\Omega - \omega_*, \quad (M_5), \tag{2.8}$$

$$2\ddot{\delta} + {}^1/_2\Omega^2\delta - {}^1/_2\,\omega_*{}^2\delta^{-3} + n\delta = j_* / \delta v, \quad (M_3), \tag{2.9}$$

$$\ddot{D} + {}^1/_2\Omega^2 D - {}^1/_2\omega_*{}^2 / D + nD = j_* / v, \quad (M_4), \tag{2.10}$$

$$\ddot{D} - 2D\;(1 + D^2)^{-2}(D\Omega - \omega_*)^2 +$$
$$+ (n + \Omega^2) D = \Omega\omega_* + j_* / v, \quad (M_5). \tag{2.11}$$

Equations (2.7) and (2.9) differ from the equations of a plane symmetric and axisymmetric beam [2] by the definition of the functions Ω and n, which enables us to describe beams with a curvilinear axis and relativistic velocities and having completely arbitrary cross section.

In what follows we consider a few solutions of (2.6)–(2.11) for a beam with a helical axis in a helical magnetic field on a circular cylinder, so that k, $\varkappa$, Ω_l, Ω_s, and Ω_q are taken to be constant.

2.1. For a beam with constant cross section (D = 1 in M_3) Eqs. (2.9) and (1.12), (1.18) lead, respectively, to the equations determining the velocity v (l) in both nonrelativistic and relativistic cases:

$${}^1/_2\dot{v} = C_* - F,$$

$$2F \equiv \Omega_*{}^2 v^2 + k^2 v^4 - {}^4/_3 k\,\Omega_q v^3 - 2j_* v, \tag{2.12}$$

$$c^2(\gamma')^2 = C_* - F \equiv$$
$$C_* + (j_* / c)\; 2\operatorname{arc\,tg} v / c - (kv^2 / c\gamma)^2 -$$
$$- (\Omega_l{}^2 - \omega_*{}^2)\ln\gamma + (\Omega_s{}^2 + \Omega_q{}^2)\gamma^{-2} +$$
$$+ 2\,\Omega_q k\;(c \operatorname{arc\,tg} v / c - v\gamma^{-2}),$$
$$\Omega_*{}^2 \equiv {}^1/_2(\Omega_l{}^2 - \omega_*{}^2) + \Omega_s{}^2 + \Omega_q{}^2,$$
$$c\,(\gamma^2 - 1)^{1/2} \equiv v. \tag{2.13}$$

It can be seen from (2.14) that the velocity is limited by a value determined by the equation F = C_*. In the case of C_* = 0 the periodic solution with amplitude v_* (Fig. 2) is obtained,

$$v_* \equiv (\Omega_* / k)\, u, \quad u^3 + u + 2\lambda u^2 = i_*,$$

$$i_* \equiv 2j_* k\;\Omega^{-3}, \quad \lambda \equiv -{}^2/_3\Omega_q / \Omega_*.$$

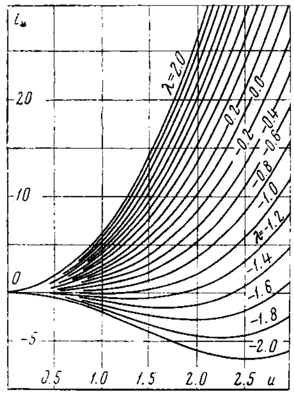

Fig. 2

In particular the periodic solution

$$v = (2j_* k^{-2})^{1/3}(\sin^3/_2\, kl)^{2/3}, \quad \Omega_s = \Omega_q = 0,$$

$$\Omega_l = \omega_* \tag{2.14}$$

satisfies the boundary conditions in the plane l = 0 at the cathode. The velocity as a function of arc length (2.14) is the same as in the exact solution for electrostatic flow over a disk [6]. However, the approximate solution (2.14) is also applicable for flow along a helical magnetic field strength line. The solution for a relativistic beam with conditions (2.14) has a period l_* and an amplitude γ_* (Fig. 3),

$$\frac{1}{2}kl_* = \int\limits_1^{\gamma_*} \frac{d\gamma}{\sqrt{\alpha_* 2\operatorname{arc\,tg} v / c - v^4 c^{-4}\gamma^{-2}}}, \quad \alpha_* \equiv \frac{j_*}{k^2 c^3},$$

$$(\gamma_* - 1 / \gamma_*)^2 = \alpha_*\{4\operatorname{arc\,tg}[\gamma_* + (\gamma_*{}^2 - 1)^{1/2}] - \pi\}. \tag{2.15}$$

For a beam with a constant width along s (D = 1 in M_1, the equation for v follows from (2.7), (1.12):

$${}^1/_2(v')^2 = C_* + j_* v -$$
$$-(\Omega_l{}^2 + \Omega_s{}^2 + \Omega_q{}^2 - \Omega_l\omega_*){}^1/_2\, v^2 + {}^2/_3 v^3(\varkappa\Omega_l +$$
$$+ k\,\Omega_q - \varkappa\omega_*) - {}^1/_2(k^2 + \varkappa^2)\, v^4, \tag{2.16}$$

which has the same form as in (2.12), but is a function of the torsion of the axis. A similar result is also obtained in the relativistic case.

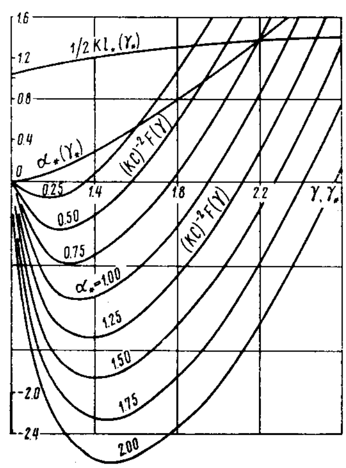

Fig. 3

2.2. For a beam with constant velocity, periodic solutions are in general obtained from (2.7), (2.9)–(2.11),

$$D = (\Omega\omega_* + j_* / v)\,\omega_1^{-2} + A_* \sin \omega_1\tau + B_* \cos \omega_1\tau,$$
$$\omega_1^2 \equiv n + \Omega^2, \quad M_1,\ M_2,$$
$$\dot{\delta}^2 + {}^1/_2\,(n + {}^1/_2\Omega^2)\,\delta^2 + {}^1/_4\omega_*{}^2\delta^{-2} =$$
$$= C_* + j_* / v \ln \delta, \quad M_3, \tag{2.17}$$

$$\dot{D}^2 = C_* + [2j_* / vD + \omega_*{}^2 \ln D - (n + {}^1/_2\Omega^2)\,D^2], \quad M_4, \tag{2.18}$$

$$\dot{D}^2 = C_* + 2\,(\Omega\omega_* + j_* / v)\,D -$$
$$-(n + \Omega^2)\,D^2 + 2\,(\Omega^2 - \omega_*{}^2 +$$
$$+ 2\Omega\omega_*\,D)\,(1 + D^2)^{-1} +$$
$$+ 2\Omega^2 \ln (1 + D^2) - 4\Omega\omega_* \operatorname{arc\,tg} D, \quad M_5. \tag{2.19}$$

Equation (2.17) has the same form as the equation for the boundary pulsations of an axisymmetric beam with a rectilinear axis in a uniform magnetic field [7].

The parameters of the periodic solutions (2.17) are calculated in [7]. The structure of Eqs. (2.18), (2.19) is qualitatively similar to (2.17).

2.3. Periodic solutions can also be constructed for a beam with a constant density ρ_*:

$$v = v_* / D, \qquad \rho_* \equiv j_* / v_*,$$
$$D = b^2 + (b^4 - a^2)^{1/2} \sin \omega\tau. \tag{2.20}$$

The solution (2.20) is obtained from (1.12), (2.7), (2.9), (2.10) for a nonrelativistic beam if the deformations M_1, M_2, M_3, M_4 are given the following values, respectively:

$$\varkappa v_*\omega_* + (k^2 + \varkappa^2)\,v_*{}^2 = a^2\omega^2,$$
$${}^1/_2\Omega_l\omega_* + v_*\,(\varkappa\Omega_l + k\Omega_q) = 2b^2\omega^2,$$
$$\Omega_l{}^2 + \Omega_s{}^2 + \Omega_q{}^2 - \rho_* = \omega^2, \quad M_1,\ M_2,$$
$$k^2v^2{}_* - {}^1/_4\omega^2{}_* = a^2\omega^2, \quad v_*k\Omega_q = 2b^2\omega^2,$$
$${}^1/_2\Omega_l{}^2 + \Omega_s{}^2 + \Omega^2{}_q - \rho_* = \omega^2,\ M_4. \tag{2.21}$$

The coefficients in the case of M_3 differ from the coefficients of M_4 by a factor 4/3 in the left-hand side. The following equation is obtained from (1.12), (2.11) in the case of M_5:

$$\dot{D}^2 = -\omega^2\,(a^2 - 2b^2\,D + D^2) +$$
$$+ 2D^2\,(1 + D^2)^{-2}\,(D\Omega_l - \omega_* - 2\varkappa v_*)^2,$$

which also describes the periodic variation of D. Here ω, a, and b are taken in accordance with (2.21).

2.4. The solutions of paragraph 2.1 may be used for constructing an electron gun with a curvilinear beam, and the solution of paragraphs 2.2, 2.3 for constructing a channel which focuses an extended periodic beam, if we specify a potential φ outside the beam, where φ satisfies Laplace's equation

$$\varepsilon^2\,[r^2 / g\,(\varphi_{,l} - \varkappa\varphi_{,\theta})]_{,l} + (g\varphi_{,r})_{,r} +$$
$$[{}^1/g\,(\sigma^2\varphi_{,\theta} - \varepsilon^2\varkappa r^2\varphi_{,l}]_{,\theta} = 0,$$
$$s \equiv r\cos\theta, \quad q \equiv r\sin\theta,$$
$$g \equiv r\,(1 - \varepsilon kr\cos\theta), \quad \sigma^2 \equiv g^2r^{-2} + \varepsilon^2\varkappa^2r^2. \tag{2.22}$$

Here l, r, and θ are quasi-cylindrical coordinates.

Representing the potential outside the beam in the form (1.9), (1.12).

$$\varphi = \varepsilon^{-2}U + \varepsilon^{-1}r\,(E_s\cos\theta + E_q\sin\theta) +$$
$$r^2\,(F_0 + F_c\cos 2\theta + F_s\sin 2\theta) + \psi,$$
$$F_0 \equiv {}^1/_4\,(3k^2v^2 + \Omega_q{}^2 + \Omega_s{}^2 - 3\Omega_qkv + 2\Omega_l\varkappa v - 2\varkappa^2v^2),$$
$$F_c \equiv {}^1/_4\,(3k^2v^2 + \Omega_q{}^2 - \Omega_s{}^2 - 3\Omega_qkv + 2\Omega_l\varkappa v),$$
$$F_s \equiv k\Omega_sv - {}^1/_2\Omega_s\,\Omega_q, \tag{2.23}$$

the following approximate equation may easily be obtained from (2.22):

$$(r\psi_{,r})_{,r} + r^{-1}\psi_{,\theta\theta} = \varepsilon k\,[(\cos\theta\psi_{,\theta})_{,\theta} + \cos\theta\,(r^2\psi_{,r})_{,r}] -$$
$$- \varepsilon^2r\,[\psi_{,ll} - (\varkappa\psi_{,\theta})_{,l} - \varkappa\psi_{,l\theta} + \varkappa^2\psi_{,\theta\theta}] - nr - \varepsilon r^2\,(n_1\cos\theta +$$
$$+ n_2\sin\theta) - \varepsilon^2r^3\,(n_3\cos 2\theta + n_4\sin 2\theta + n_5),$$
$$n_1 \equiv (kU')' + E_s'' - (\varkappa E_q)' - \varkappa^2E_s - \varkappa E'_q - k\,(6F_0 + 2F_c),$$
$$n_2 \equiv k\varkappa U' + E_q'' + (\varkappa E_s)' - \varkappa^2E_q + \varkappa E_s' - 2kF_s,$$
$$n_3 \equiv {}^1/_2\,(k^2U')' + {}^1/_2\,(kE_s')' - {}^1/_2\,(\varkappa kE_q)' - k\varkappa^2E_s -$$
$$- \varkappa kE_q' + F_c'' - 2\,(\varkappa F_s)' - 2\varkappa F_s' - 4\varkappa^2F_c,$$
$$n_4 = \varkappa k^2U' + {}^1/_2\,(kE_q')' + {}^1/_2\,(k\varkappa E_s)' - k\varkappa^2\,E_q + k\varkappa E_s' +$$
$$+ F_s'' + 2\,(\varkappa F_c)' + 2\varkappa F_c' - 4\varkappa^2F_s,$$
$$n_5 \equiv {}^1/_2\,(k^2U')' + {}^1/_2\,(kE_s')' - {}^1/_2\,(\varkappa kE_q)' + F_0'' \tag{2.24}$$

where terms of order ε^3 have been omitted. The solution of (2.24) which satisfies the boundary conditions on the surface $r = f$ of a circular beam with deformation M_3 has the form

$$\psi = W + B\ln r - {}^1/_4nr^2 -$$
$$- \varepsilon{}^1/_4\,\{{}^1/_2\,(r^3 - 2f^2r + f^4 / r)\,(n_0\cos\theta +$$
$$+ n_2\sin\theta) - (2r\ln r / f - r + f^2 / r)\,kB\cos\theta\} -$$
$$- \varepsilon^2\,{}^1/_4\,\{{}^1/_4\,(r^4 - f^4 - 4f^4\ln r / f)\,(n_5 + {}^1/_4n'' + {}^9/_8k^2n +$$
$$+ {}^3/_4kn_1) - (r^2 - f^2 - 2f^2\ln r / f)\,(W'' - {}^1/_2k^2B\ln f -$$
$$- B'' - {}^1/_4f^2\,kn_0) +$$
$$+ [(r^2 - f^2)\ln r - 2f^2\ln (r / f)\ln f]\,({}^1/_2k^2B -$$
$$- B'') - {}^1/_6\,(2r^4 - 3f^2r^2 + f^2r^{-2})\,[(n_3 + {}^5/_8kn_0)\cos 2\theta +$$
$$+ (n_4 + {}^5/_8kn_2)\sin 2\theta] - {}^1/_8kf^2\,(1 - {}^1/_2r^2f^{-2} - {}^1/_2f^2r^{-2})\times$$
$$\times [(f^2n_0 + 2kB)\cos 2\theta + f^2n_2\sin 2\theta]\},$$
$$n_0 \equiv n_1 + {}^3/_2kn, \quad B \equiv {}^1/_2r_*{}^2j_* / v,$$
$$W \equiv {}^1/_2B\,(1 - \ln f^2), \quad f \equiv r_*\delta. \tag{2.25}$$

The solution (2.23), (2.25) has the accuracy of the paraxial approximation $\varepsilon_*{}^3$ in a tube which is $\varepsilon_*{}^{-0.4}$ times wider than the beam, since an additional two terms in the expansion with respect to ε have been taken into account in (2.25).

3. Beam pulsations in a narrow cavity. Let us now consider beam pulsations between two electrodes (Fig. 4) in two cases: plane (l, s) and cylindrical (l, r), for which Eqs. (1.13)–(1.15) assume the form

$$s_{,\tau\tau} + \Omega^2s + \Omega P = -ns + J/v - Q \equiv \psi_{,s},$$
$$\rho v = (s_{,J})^{-1}, \tag{3.1}$$

$$r_{,\tau\tau} + {}^1/_4\,\Omega^2r - P^2r^{-3} =$$
$$= (J / v - Q)\,(2\pi r)^{-1} - {}^1/_2nr \equiv \psi_{,r}, \tag{3.2}$$

$$r^2 \equiv s^2 + q^2, \quad P = P\,(J),$$
$$r = r\,(\tau,\ J), \quad 2\pi\rho vr = (r_{,J})^{-1}. \tag{3.3}$$

Here J is the stream function, P is the component of the generalized momentum in the direction of the cyclical coordinate q or θ. There are many papers with the calculations of the symmetrical pulsations of the beam boundary within the framework of the equations of paraxial optics [8], which differ from (3.1), (3.2), apart from the definition of Ω and n, by the ab-

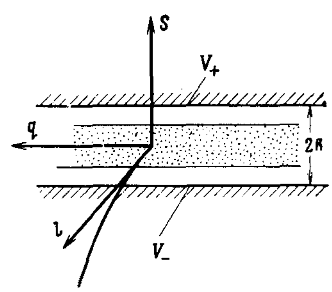

Fig. 4

sence of the term Q. The function Q appears as the result of the asymmetry of the plane beam (3.1) or also as the result of the presence of the central electrode (rod) in the axisymmetric case (Fig. 5) and its physical meaning is that of a moving charge density induced by the beam on the electrodes. For $Q \neq 0$ Eqs. (3.1), (3.2) are integrodifferential equations and do not have degenerate solutions, while the density ρ is markedly nonuniform across the beam.

3.1. Let two cylindrical electrodes be situated at distances $s = \pm b$, $b \sim a_*$ from a plane axial curve ($\varkappa = 0$) and have constant potentials V_+, V_-:

$$V_{\pm} = \varepsilon^{-2} U \pm \varepsilon^{-1} b E_s,$$
$$E_s = k v^2 - \Omega_q v, \; (v\Omega_s = E_q = 0). \tag{3.4}$$

The following expression is obtained for the potentials $\varphi_{\pm}$ in the gaps between the beam and the electrodes, with the accuracy of the paraxial approximation:

$$\varphi_{\pm} = \varepsilon^{-2} U + \varepsilon^{-1} E_s s +$$
$$+ {}^1/_2 k E_s (s^2 - b^2) + B_{\pm} [s - (\pm b)]. \tag{3.5}$$

When the external field (3.5) is matched with the internal field (1.9), (1.12) on the boundaries of the beam $s = s_{\pm}$ we have

$$B_+ = J_* / v - Q, \; Q = J_* (2vb)^{-1} (b - \langle s \rangle) = -B_-,$$
$$\langle s \rangle \equiv 1/J_* \int^{*} s dJ.$$

Here J_* is the total beam current, $\langle s \rangle$ is the centre of gravity of the beam. It follows from (3.1) that for $\langle s \rangle$

$$\langle \ddot{s} \rangle + \omega^2 \langle s \rangle = -\Omega \langle P \rangle,$$
$$\omega^2 \equiv \Omega^2 + n - J_* (2vb)^{-1}.$$

Let ω, Ω, and v be constants. It then follows from (3.1) that

$$s = \langle s \rangle + A(J) \cos \omega_2 \tau + B(J) \sin \omega_2 \tau,$$
$$\omega_2 \equiv (\Omega^2 + n)^{1/2},$$
$$\langle s \rangle = -\Omega \langle P \rangle \omega^{-2} + a_* \sin \omega\tau + b_* \cos \omega\tau.$$

It is clear that in addition to the symmetric pulsations of frequency ω_2 there is the additional oscillation of the beam centre with the shifted frequency ω. The condtion $\omega = 0$ determines the limiting beam current $2vb(\Omega^2 + n)$ in excess of which the beam centre oscillations become aperiodic.

3.2. The charge density on the rod is determined as follows for an axisymmetric beam with a rectilinear axis:

$$2\gamma Q = -4\pi u + 2\gamma J_* / v -$$
$$- \frac{1}{v} \int^{*} \ln (r / R_-)^2 \, dJ, \; \gamma \equiv \ln R_+ / R_- . \tag{3.6}$$

Here u is the potential difference between the electrodes, R_+, R_- are the radii of the electrodes (Fig. 5). Let u, v, and Ω be constant, and let $n = 0$. For these conditions the integrodifferential equation (3.2), (3.6) has an accuracy ε_*^2. A solution can be found for departures of the order ε_* from the equilibrium state,

$$r = R(J) + \frac{\varepsilon f(\tau)}{R}, \; 2\pi v \left(\frac{\Omega^2 R^2}{4} - \frac{P^2}{R^2} \right) =$$
$$= J - J_* + \frac{2\pi u v}{\gamma} + \frac{1}{\gamma} \int \ln \frac{R}{R_-} \, dJ, \tag{3.7}$$

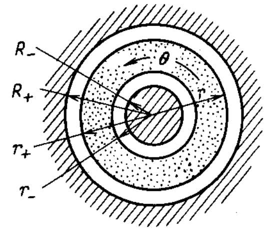

Fig. 5

$$f_{,\tau\tau} + \omega_0^2 f = (2\pi\gamma v)^{-1} \int^{*} f R^{-2} dJ,$$
$$\omega_0^2 = {}^1/_2 \Omega^2 + 2P^2 R^{-4}. \tag{3.8}$$

Equation (3.8) describes oscillations f of frequency ω_0, on which there are superimposed oscillations with the shifted frequency ω,

$$\frac{1}{\omega_0^2 - \omega^2} (c_* \cos \omega\tau + b_* \sin \omega\tau), \; \int_0^{J_*} \frac{dJ}{R^2 (\omega_0^2 - \omega^2)} = 2\pi v \gamma. \tag{3.9}$$

Curves are given for the solution of Eqs. (3.7), (3.9) in the case $P = P_* = \mathrm{const}$ in Fig. 6, where families of curves $\gamma(\nu, \mu, \lambda)$ are shown for $\mu = 2$ and $\lambda = 1$:

$$\gamma = {}^1/_4 \nu^2 (1 - \nu^2)^{-1} \{ \ln [\mu^4 (1 - \nu^2) \lambda + 1] -$$
$$- \ln [(1 - \nu^2) \lambda + 1] \} + \ln \mu,$$
$$\mu \equiv r_+ / r_-, \quad \lambda \equiv \Omega^2 r_-^4 / 4P_*, \quad \nu \equiv 2\omega^2 \Omega^{-2}.$$

Regardless of the form of the solution considerable nonlinearity remains in the problem, associated with the fact that the amplitude of the oscillations depend on J through the frequency ω_0 and the initial conditions. This nonlinearity leads to the intersection of electron trajectories at the instant $f_{,J} = 0$, which determines the limits of applicability of the solution.

Note 1. It is interesting to note that the coaxial equation for a beam which is symmetric relative to a rectilinear axis and has an appreciable nonstationary axial velocity and magnetic field [$v = v(t, l)$, $\Omega = \Omega(t, l)$] is of the form (3.2) when J/v is replaced by $J/l_{,\lambda}$. In this case

$$n \equiv U_{,ll}, \quad U = U(t, l), \quad \lambda_{,t} = -v\lambda_{,l}, \quad 2\pi r\rho = (r_{,J} l_{,\lambda})^{-1}.$$

As distinct from the steady-state beam the paraxial family of trajectories given here depends markedly on the integral of motion of an electron with axial velocity v: $r = r(t, \lambda, J)$. In particular there are also degenerate solutions of the form (2.1)–(2.4), where α, β, and ν are already functions of t and λ and a dot denotes a partial derivative with

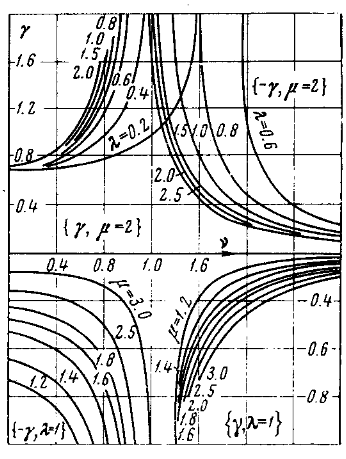

Fig. 6

respect to t. Any attempt to construct a paraxial equation for a nonsteady-state beam with a curvilinear axis necessitates a more general approach, which takes into account the nonstationarity of conditions on the axis and the nonstationarity of the axis itself.

Note 2. It can be shown that degenerate solutions satisfy zero velocity emission conditions on curvilinear cathodes also.

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